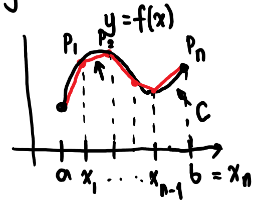
Lecture 35

Wednesday, November 30, 2016 9:07 AM

8.1 Arc Length

GOAL Find the length of curve C defined by the equation $y = f(x)$, f is continuous & $a \le x \le b$.



$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

We will require that f' be continuous.

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

(Proof in the book).

Ex Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1+(-\tan x)^2}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln\left|\sec 0 + \tan 0\right|$$

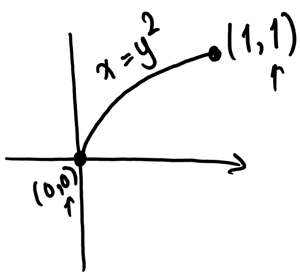
$$= \ln|\sqrt{2}+1| - \ln|1+0|$$

$= \ln(\sqrt{2}+1)$

- If the curve C has eqn $x = g(y)$, where $g'(y)$ is continuous and $c \le y \le d$, then the length L of the curve C is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Ex Find the length of the arc of the parabola $x = y^2$ from $(0,0)$ to $(1,1)$.



$$\frac{dx}{dy} = 2y$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + (2y)^2}\, dy$$

$u = 2y \qquad du = 2\,dy \Rightarrow dy = \frac{du}{2}$

$y = 0,\ u = 0$

$y = 1,\ u = 2$

$$\int_0^2 \sqrt{1+u^2}\, du \qquad , \ u = \tan\theta$$

$$du = \sec^2\theta\, d\theta$$

α

$$= \frac{1}{2}\int_0^{\alpha} \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$

$du = \sec^2\theta\,d\theta$

$u = 0,\ \tan\theta = 0 \Rightarrow \theta = 0$

$u = 2,\ \tan\theta = 2 \Rightarrow \theta = \arctan(2) = \alpha$

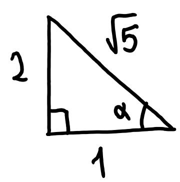
$$= \frac{1}{2}\int_0^{\alpha} \sec\theta \cdot \sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\alpha} \sec^3\theta\,d\theta$$ → Lecture Notes

$$= \frac{1}{2}\left[\frac{1}{2}\left[\sec\theta \cdot \tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\alpha}\right]$$

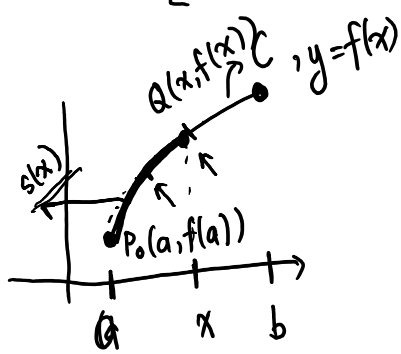
$$= \frac{1}{4}\left[\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha| - 0\right]$$

$\alpha = \tan^{-1}(2) \Rightarrow \tan\alpha = 2$



$\sec\alpha = \frac{\sqrt{5}}{1} = \sqrt{5}$

$$= \frac{1}{4}\left[\sqrt{5}\cdot 2 + \ln\left(\sqrt{5}+2\right)\right]$$



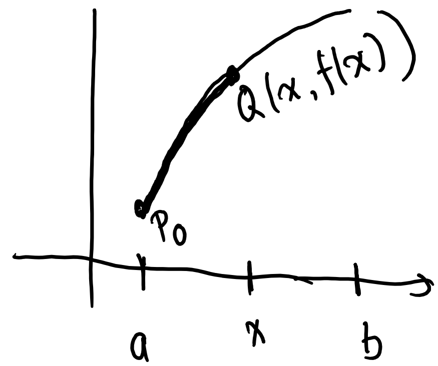
THE ARC LENGTH FUNCTION

Useful to have a function that measure the arc length of a curve from a prescribed

starting point to any other
point on the curve.

Suppose C given by $y = f(x)$, $a \le x \le b$
where f is smooth (f' continuous).

Let $s(x)$ be the distance along C between
the initial point $P_0\,(a, f(a))$ to the point
$Q(x, f(x))$. $s(x)$ is called the arc length
function,



$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} \quad \text{by FTC part I}$$

rate of change of s with respect to x is
always at least 1, and equals 1 when
$f'(x)$, slope of the curve, is 0.

$$ds = \underbrace{\sqrt{1 + (f'(x))^2}\, dx} \Rightarrow L = \int ds$$

Ex Find the arclength function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1,1)$ as the starting point.

$f(x) = x^2 - \frac{1}{8}\ln x$

$f'(x) = 2x - \frac{1}{8} \cdot \frac{1}{x}$

$$s(x) = \int_a^{x} \sqrt{1 + (f'(t))^2}\, dt$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2$$

$$= 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2$$

Then, $\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$

Then the arc length function, $s(x)$ is given by

$$s(x) = \int_1^x \sqrt{1+(f'(t))^2}\,dt$$

$$= \int_1^x 2t + \frac{1}{8t}\,dt$$

$$= \left[t^2 + \frac{1}{8}\ln t\right]_1^x$$

$$= x^2 + \frac{1}{8}\ln x - \left(1^2 + \frac{1}{8}\ln 1\right)$$

$$= x^2 + \frac{1}{8}\ln x - 1$$